STABILITY OF AN INCOMPRESSIBLE FLUID ACTED ON BY SURFACE-TENSION FORCES. CASE OF A DOUBLY CONNECTED EQUILIBRIUM SURFACE

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The stability of the equilibrium position of a volume of incompressible fluid is considered; it is bounded by the rigid walls of the vessel and two equilibrium surfaces. The stability conditions are expressed in terms of parameters determined for each of the surfaces by independent solution of the eigenvalue problem. The stability of an arbitrary volume of incompressible fluid having spherical segments as the two equilibrium surfaces is investigated as an example.

1. Let a volume Q of an incompressible fluid bounded by the rigid walls of a vessel S and two interfaces Σ_1, Σ_2 be located in a field of mass forces with potential Π. We assume for simplicity that Σ_1 and Σ_2 are free surfaces. Let σ_1, σ_2 be the coefficients of surface tension on Σ_1 and Σ_2, and let α_1, α_2 be the differences in the coefficients of surface tension on the rigid body–fluid and rigid body–gas boundaries in the neighborhood of the contours L_1 and L_2 formed by the intersection of the free surfaces and S. Then the potential energy of the system takes the form

$$U = \sigma \int\limits_{\Sigma_1} d\Sigma_1 + \sigma_2 \int\limits_{\Sigma_2} d\Sigma_2 + \alpha_1 \int\limits_{L_1} dl_1 + \alpha_2 \int\limits_{L_2} dl_2 + \int\limits_{Q} \Pi dQ \tag{1.1}$$

The necessary conditions for equilibrium of the volume Q are obtained by setting the first variation of the potential energy equal to zero [2, 3]. We assume further that the problem of finding the equilibrium form has been solved. The stability of the given equilibrium position is tested by determining the sign of the second variation of the potential energy. According to [4], a necessary condition for stability is that the second variation not be negative; a sufficient condition is

$$\delta^2 U > 0 \tag{1.2}$$

We shall make use of the expression obtained in [1] for the second variation of the energy of the equilibrium surface Σ. Let n_i ($i = 1, 2$) be the unit normal, exterior to the region Q, to the surface Σ_i, and let $N_i(\xi)$ be the small deviation of the point $\xi \in \Sigma_i$ along the direction n_i. The second variation in the energy U_i of the i-th surface has the form [1] of a quadratic functional in the perturbation $N = N_i$ ($i = 1, 2$),

$$\delta^2 U_i / \sigma_i = \int\limits_{\Sigma_i} (-\Delta_i N + \tau_i N)\, N\, d\Sigma_i + \int\limits_{L_i} (\chi_i N + \partial N / \partial e_i)\, N\, dl_i \tag{1.3}$$

Here Δ_i is a Laplace operator [5] on the surface Σ_i, while the functions τ_i, χ_i are equal to the following expressions [1]:

$$\tau_i(\xi) = \frac{1}{\sigma_i} \frac{\partial \Pi}{\partial n_i} - 4H_i^2(\xi) + 2K_i(\xi), \quad \xi \in \Sigma_i \tag{1.4}$$

$$\chi_i(\xi) = \frac{\varkappa_i(\xi)\cos\gamma_i - \eta_i(\xi)}{\sin\gamma_i} (\sin\gamma_i \neq 0), \quad \xi \in L_i \tag{1.5}$$

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where $H_i(\xi)$ is the mean curvature, and $K_i(\xi)$ is the Gaussian curvature [5] at the point $\xi \in \Sigma_i$; γ_i is the edge angle, $\gamma_i = \alpha_i/\sigma_i$, $\varkappa_i(\xi)$ is the curvature of the normal cross section of the surface Σ_i along the tangent to it, directed along the exterior normal to L_i (we let $\mathbf{e}_i$ represent the corresponding unit vector); $\eta_i(\xi)$ is the similarly defined curvature of the normal cross section of the surface S at the point $\xi \in L_i$.

We consider the following eigenvalue problem on the surfaces Σ_i (i = 1, 2):

$$A_i u \equiv \sigma_i(-\Delta_i + \tau_i)\; u(\xi) = \lambda u(\xi),\;\; \xi \in \Sigma_i$$
$$\chi_i u(\xi) + \partial u / \partial e_i = 0,\;\; \xi \in L_i \tag{1.6}$$

It may be shown [6] that each operator A_i (i = 1, 2) is self-adjoint on the set D_i of functions twice continuously differentiable on Σ_i that satisfy the boundary condition (1.6) for the i-th problem on the contour L_i. Thus, the eigenvalues of each problem of (1.6) are real numbers, while the eigenfunctions corresponding to the various eigenvalues are orthogonal on the corresponding surface Σ_i. Here the set of eigenvalues of the i-th problem of (1.6) takes the form of a sequence [6] bounded from below and converging to k $+\infty$.

Let φ_k, ν_k (k = 1, 2, ...) be the eigenfunctions and eigenvalues of (1.6) on Σ_1, and ψ_k, $\varkappa_k$ those on Σ_2. With no loss of generality we may take the orthogonal systems of functions $\{\varphi_k\}$, $\{\psi_k\}$ to be normalized and the sequences of eigenvalues $\{\nu_k\}$, $\{\varkappa_k\}$ to be arranged in ascending order,

$$\nu_1 \leqslant \nu_2 \leqslant \ldots,\; \varkappa_1 \leqslant \varkappa_2 \leqslant \ldots \tag{1.7}$$

Any perturbations $N_1 \in D_1$, $N_2 \in D_2$ of the surfaces Σ_1, Σ_2 are representable as converging series

$$N_1 = \sum_{k=1}^{\infty} a_k \varphi_k, \quad N_2 = \sum_{k=1}^{\infty} b_k \psi_k \tag{1.8}$$

Using (1.3), (1.8) to calculate $\sigma^2 U_1$, $\sigma^2 U_2$, we find

$$\delta^2 U = \delta^2 U_1 + \delta^2 U_2 = \sum_{k=1}^{\infty} \nu_k a_k^2 + \sum_{k=1}^{\infty} \varkappa_k b_k^2 \tag{1.9}$$

The condition requiring the volume Q to be constant yields a relationship between the sequences $\{a_k\}$, $\{b_k\}$,

$$\delta Q = \int_{\Sigma_1} N_1\, d\Sigma_1 + \int_{\Sigma_2} N_2\, d\Sigma_2 = \sum_{k=1}^{\infty} v_k a_k + \sum_{k=1}^{\infty} w_k b_k = 0 \tag{1.10}$$

$$v_k = \int_{\Sigma_1} \varphi_k\, d\Sigma_1, \quad w_k = \int_{\Sigma_2} \psi_k\, d\Sigma_2 \tag{1.11}$$

We assume the perturbations N_1, N_2 to be nonzero. This condition may be taken in the form

$$\int_{\Sigma_1} N_1^2\, d\Sigma_1 + \int_{\Sigma_2} N_2^2\, d\Sigma_2 = \sum_{k=1}^{\infty} a_k^2 + \sum_{k=1}^{\infty} b_k^2 = 1 \tag{1.12}$$

2. The stability investigation reduces to checking the sign of Series (1.9) for Conditions (1.10), (1.12). Let W be the set of all pairs of sequences $\{a_k, b_i\}$, that satisfy (1.10), (1.12). According to (1.2), it is sufficient for stability of the system for (1.9) to be positive on W. If there exists a set of numbers $\{a_k, b_i\} \in W$, such that the variation $\delta^2 U$, determined in accordance with (1.9) is negative, then the system is unstable. Let us look at some cases in which the question of the sign of $\delta^2 U$ is answered trivially.

A. If $\nu_1 > 0$, $\varkappa_1 > 0$, then as a consequence of (1.7), the expression (1.9) for $\delta^2 U$ is positive on W and the system is stable.

B. Let $\nu_1 < 0$, $v_1 = 0$. We let $a_1 = 1$ and substitute zeros for all the remaining a_k, b_i. Conditions (1.10), (1.12) are satisfied. But for such a set of coefficients

$$\delta^2 U = \nu_1 < 0$$

and the system is unstable. An analogous statement holds for any negative eigenvalue if the volume (1.11) for the corresponding eigenvalue is zero.

C. The system will also be unstable if among the numbers $\{\nu_k\}$, $\{\varkappa_i\}$ there are even two that are negative [with nonzero volumes (1.11)]. The coefficients corresponding to these two eigenvalues are uniquely determined by (1.10), (1.12), if we set the remaining a_k, $b_i \equiv 0$; here (1.9) is negative.

We still must consider the case in which there is just one negative eigenvalue corresponding to an eigenfunction for which the coefficient (1.11) is nonzero. To be specific we shall assume ν_1 is negative. We find the minimum of (1.9) on the set W, considering the following auxiliary functional:

$$V = \delta^2 U + \lambda\left(1 - \sum_{k=1}^{\infty} a_k^2 - \sum_{k=1}^{\infty} b_k^2\right) + \mu\left(\sum_{k=1}^{\infty} v_k a_k + \sum_{k=1}^{\infty} w_k b_k\right) \tag{2.1}$$

where λ, μ are Lagrange multipliers.

The necessary conditions for an extremum of the functional V yield

$$a_k = -\mu v_k / (\nu_k - \lambda), \quad b_k = -\mu w_k / (\varkappa_k - \lambda) \tag{2.2}$$

In conjunction with (2.2), the normalization condition (1.12) permits us to express μ in terms of λ

$$\mu(\lambda) = \left[\sum_{k=1}^{\infty} \frac{v_k^2}{(\nu_k - \lambda)^2} + \sum_{k=1}^{\infty} \frac{w_k^2}{(\varkappa_k - \lambda)^2}\right]^{-1/2} \tag{2.3}$$

Then

$$\delta^2 U = \mu^2(\lambda)\left[\sum_{k=1}^{\infty} \frac{\nu_k v_k^2}{(\nu_k - \lambda)^2} + \sum_{k=1}^{\infty} \frac{\varkappa_k w_k^2}{(\varkappa_k - \lambda)^2}\right] \tag{2.4}$$

Allowing for (2.3), from the condition requiring conservation of the volume (1.10) we obtain an equation for the unknown value of the parameter,

$$\sum_{k=1}^{\infty} \frac{v_k^2}{\nu_k - \lambda} + \sum_{k=1}^{\infty} \frac{w_k^2}{\varkappa_k - \lambda} = 0 \tag{2.5}$$

Equation (2.5) has a countable set of roots $\{\lambda_j\}$, which alternate with the poles ν_k, $\varkappa_i$. By virtue of (1.7) we may say that

$$\lambda_1 \leqslant \lambda_2 \leqslant \lambda_3 \leqslant \ldots \tag{2.6}$$

Substituting the arbitrary root λ_j of (2.5) into (2.4), the expression for the second variation of the energy, and making use of (2.3), we may show without difficulty that

$$\delta^2 U(\lambda_j) = \lambda_j \ (j = 1, 2, \ldots) \tag{2.7}$$

Thus, from (2.7), (2.6) we have

$$\min_W \delta^2 U = \lambda_1 \tag{2.8}$$

We therefore see that the stability question is answered either by one of the conditions A, B, C or else is determined in accordance with (1.2) and (2.8) by the sign of the first root λ_1 of (2.5). In the latter case a necessary condition for stability is that the first root of Eq. (2.5) be nonnegative, while a sufficient condition is that this root be positive.

We note that if the kinematic constraints are such that on both surfaces only perturbations of the zero volume are admissible, then (2.5) splits into two equations,

$$\sum_{k=1}^{\infty} \frac{v_k^2}{\nu_k - \lambda} = 0, \quad \sum_{k=1}^{\infty} \frac{w_k^2}{\varkappa_k - \lambda} = 0 \tag{2.9}$$

The stability decision should be made on the basis of the sign of the smallest of the first roots of (2.9). There may be cases in which the first roots of (2.9) are positive, while the first root of (2.5) is negative, i.e., a system that is stable under surface perturbations of the zero volume will prove to be unstable under arbitrary perturbations $N_1 \in$

D_1, $N_2 \in D_2$ connected by Condition (1.10) (see §4).

These results may be generalized without difficulty to the case of an incompressible fluid having n equilibrium surfaces with $n > 2$.

3. Let us consider the case in which both free surfaces Σ_1, Σ_2, the external field Π, and the wetted surface S (in the neighborhood of the contours L_1, L_2) possess axial symmetry. We let Z be the axis of symmetry. We introduce the cylindrical coordinates $\{r, 2, \theta\}$. As the curvilinear coordinates on the surfaces Σ_1, Σ_2 we take the angle θ and the length of arc s measured along a meridian. We specify the meridian of surface Σ_i parametrically:

$$r = r_i(s),\ z = z_i(s),\ 0 \leqslant s \leqslant c_i\ (i = 1,2)$$

where c_i is the length of the meridian of the i-th surface.

The Laplace operator on Σ_i has the form

$$\Delta_i = \frac{1}{r_i}\frac{\partial}{\partial s}\left(r_i\frac{\partial}{\partial s}\right) + \frac{1}{r_i^2}\frac{\partial^2}{\partial \theta^2} \tag{3.1}$$

The eigenfunctions $\{\varphi_k(s, \theta)\}$, $\{\psi_k(s, \theta)\}$ of (1.6) split into the following families on Σ_1, Σ_2 ($n = 0, 1, \ldots$, $k = 1,2,\ldots$):

$$\left\{\varphi_{nk}(s)\begin{matrix}\sin\\ \cos\end{matrix} n\theta\right\}, \quad \left\{\psi_{nk}(s)\begin{matrix}\sin\\ \cos\end{matrix} n\theta\right\} \tag{3.2}$$

where $\{\varphi_{nk}\}$, $\{\psi_{nk}\}$, are the sets of eigenfunctions of the one-dimensional problems ($i = 1,2$),

$$\begin{gathered}\sigma_i\left(-\frac{1}{r_i}\frac{d}{ds}\left(r_i\frac{d}{ds}\right) + \frac{n^2}{r_i^2} + \tau_i\right)u = \lambda u \\ 0 < s < c_i,\ n = 0,\ 1,\ \ldots \\ [\chi_i u - u']_{s=0} = [\chi_i u + u']_{s=c_i} = 0\end{gathered} \tag{3.3}$$

The function τ_i is determined by (1.4) and the parameter χ_i by (1.5), or by the condition for boundedness of the solution if the corresponding end points of the meridian lie on the z axis.

The functions φ_{nk}, ψ_{ni} (3.2) correspond to the eigenvalues ν_{nk}, $\varkappa_{ni}$ of the problems (3.3); the numbering is such that the numbers $\{\nu_{nk}\}$, $\{\varkappa_{ni}\}$ satisfy (1.7) for any fixed value $n (=0, 1, \ldots)$.

Clearly, when $n \geq 1$ all the volumes v_{nk}, w_{ni} (1.11) will equal zero. Thus, for instability of the system it is sufficient (case B, §2) for just one of the numbers ν_{nk}, $\varkappa_{ni}$ ($n \geq 1$) to be negative. It has been shown in [1] that when $n > 1$ the inequalities

$$\nu_{n1} > \nu_{11},\ \varkappa_{n1} > \varkappa_{11}$$

are satisfied.

Then, when (1.7) is taken in account, the necessary conditions for stability under perturbations having nonzero ($n \geq 1$) harmonics in the peripheral direction take the form

$$\nu_{11} \geqslant 0,\ \varkappa_{11} \geqslant 0 \tag{3.4}$$

In general, the method used in §2 should be employed to investigate stability under axisymmetric perturbations ($n = 0$).

We note that for a field of mass forces having potential $\Pi = Bz$ (B is a constant, z is the value of the coordinate along the axis parallel to the field) everything we have said also applies to surfaces Σ_1, Σ_2 whose axes of symmetry do not coincide but are parallel to the direction of the external field. In case of weightlessness the axes of symmetry may be arbitrarily oriented.

4. Let us consider a model problem. Under conditions of weightlessness let a volume Q of incompressible fluid be bounded by rigid walls S and have two free surfaces Σ_1, Σ_2. We assume that the surface S is axisymmetric in the neighborhood of the contours L_1, L_2 (see §1) and that the pressures P_1, P_2 on Σ_1, Σ_2 are constant. Then Σ_1, Σ_2 will have the form of

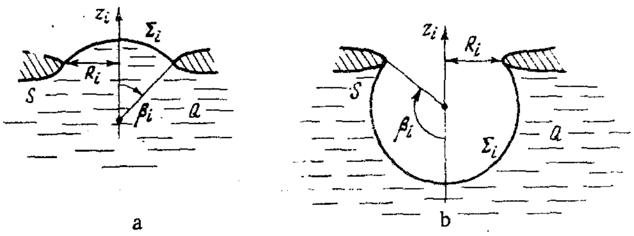

Fig. 1

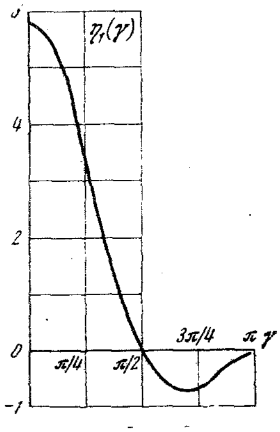

Fig. 2

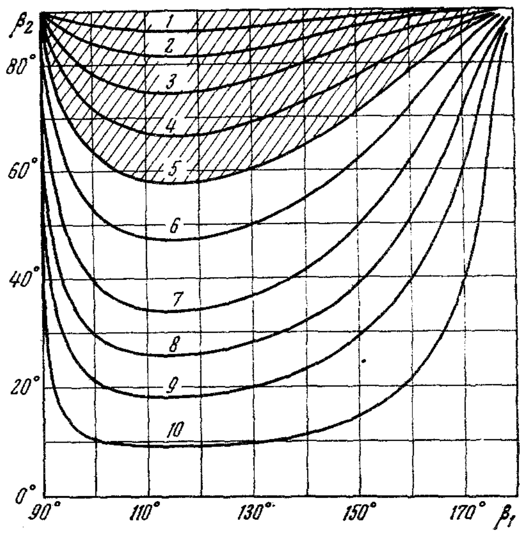

Fig. 3

spherical segments. We let β_1, β_2 be the half-angles of the segments and R_1, R_2, the radii of their bases. The necessary condition for equilibrium of the volume is

$$P_1 \pm \frac{\sigma_1 \sin \beta_1}{R_1} = P_2 \pm \frac{\sigma_2 \sin \beta_2}{R_2} \tag{4.1}$$

The plus sign in (4.1) corresponds to the case in which the i-th meniscus is convex with respect to the region Q (Fig. 1a) and the minus sign to the i-th meniscus, concave (Fig. 1b).

Assuming that the fluid ideally wets the surface S of the rigid walls up to the contours L_1, L_2, we consider the stability of an arbitrary equilibrium system of two spherical segments having parameters $\{\sigma_i, R_i, \beta_i\}$.

On the surface Σ_i the problem (3.3) has the form (i = 1, 2)

$$-\frac{\sigma_i \sin^2 \beta_i}{R_i^2}\left(\frac{d^2}{d\alpha^2} + \operatorname{ctg}\alpha \frac{d}{d\alpha} - \frac{n^2}{\sin^2\alpha} + 2\right) u = \lambda u$$
$$0 < \alpha < \beta_i, \quad n = 0,1 \tag{4.2}$$
$$|u(0)| < +\infty, \quad u(\beta_i) = 0$$

Here the boundary condition at point β_i follows from the complete wettability.

We note that (4.2) has the same exact form for both convex and concave menisci of Σ_i, while the volumes v_k, w_j (1.11) for the eigenfunctions of the problem (4.2) enter into (2.5) as the second power. Thus, the stability of the system $\{\Sigma_1, \Sigma_2\}$ of spherical segments does not depend on the meniscus convexity-concavity combination, but is determined solely by the quantities $\{\sigma_i, R_i, \beta_i\}$.

Let us consider the following auxiliary problem ($0 < \gamma < \pi$):

$$-\sin^2\gamma\left(\frac{d^2}{d\alpha^2}+\operatorname{ctg}\alpha\frac{d}{d\alpha}-\frac{n^2}{\sin^2\alpha}+2\right)u=\eta u$$
$$0<\alpha<\gamma,\ n=0,1\quad |u(0)|<+\infty,\ u(\gamma)=0 \tag{4.3}$$

For $\gamma = \beta_i$ and fixed n, the eigenvalues $\lambda_{ij}(\beta_i)$ of (4.2) and $\eta_j(\gamma)$ of (4.3) are connected by the following relationship ($i = 1, 2$; we omit the index n):

$$\lambda_{ij}(\beta_i)=\frac{\sigma_i}{R_i^2}\eta_j(\beta_i),\qquad j=1,2,\ldots \tag{4.4}$$

Let us check to see that the necessary conditions (3.4) for stability of the system with respect to nonzero harmonics ($n \geq 1$) of the perturbations are satisfied. The first eigenfunction of (4.3) for $n = 1$ is an associated Legendre function of the first kind [7], P_{q*}^1 $(\cos\alpha)$, where q_* is the first root of the equation $P_q(\cos\gamma) = 0$. Calculations for various values of the angle $\gamma \in (0, \pi)$ show that $q_* > 1$. Thus, the first eigenvalue of (4.3) for $n = 1$,

$$\eta_{11}(\gamma)=(q_*(q_*+1)-2)\sin^2\gamma$$

is positive for $0 \leq \gamma < \pi$. Taking (4.4) into account, we see that the necessary stability conditions (3.4) are satisfied.

Let us consider the influence of axisymmetric perturbations. For $n = 0$ the eigenfunctions of (4.3) are Legendre functions of the first kind [7], $P_{\mu_k}(\cos\alpha)$, where μ_k are sequential roots of the equation

$$P_\mu(\cos\gamma)=0 \tag{4.5}$$

The corresponding eigenvalues of (4.3) are

$$\eta_k(\gamma)=(\mu_k(\mu_k+1)-2)\sin^2\gamma \tag{4.6}$$

By going to the limit we can show that for $\gamma = 0$ the eigenfunctions of (4.3) are the Bessel functions $J_0(\xi_k r)$, where $r \in (0, 1)$, and the ξ_k are roots of the equation $J_0(\xi) = 0$ [8]; here $\eta_k(0) = \xi_k^2$. For $\gamma = \pi/2$ the eigenfunctions of (4.3) are Legendre polynomials [7], $P_{2k-1}(\cos\alpha)$, i.e., $\eta_k(\pi/2) = 2k(2k - 1) - 2$.

For arbitrary $\gamma \in (0, \pi)$, Eq. (4.5) may be solved by digital computer. Calculations have shown that the eigenvalue $\eta_1(\gamma)$ is positive for $0 \leq \gamma < \pi/2$ and negative for $\pi/2 < \gamma < \pi$ (Fig. 2). The remaining eigenvalues $\eta_k(\gamma)$ are positive for $0 \leq \gamma < \pi$.

Depending on the values of the half-angles β_1, β_2 of the spherical segments Σ_1, Σ_2 we distinguish three types of equilibrium forms: I) $\beta_1, \beta_2 < \pi/2$, II) $\beta_1, \beta_2 > \pi/2$, III) one angle greater than $\pi/2$ and the other less than $\pi/2$. Taking into account (4.4) and cases A, C of § 2, from the way in which the eigenvalue η_1 depends on the angle γ we conclude that the type-I equilibrium forms are stable, while the type-II equilibrium forms are unstable for any values of the parameters σ_i, R_i ($i = 1, 2$).

We still must investigate the stability of the type-III equilibrium forms. We let $g_k(\gamma)$ represent the volume of the normalized eigenfunction of (4.3) for $n = 0$,

$$g_k(\gamma)=\frac{1}{\sin\gamma}\int_0^\gamma P_{\mu_k}(\cos\alpha)\sin\alpha\,d\alpha\left[\int_0^\gamma(P_{\mu_k}(\cos\alpha))^2\sin\alpha\,d\alpha\right]^{-1/2} \tag{4.7}$$
$$0<\gamma<\pi,\ k=1,2,\ldots$$

Let $v_k(\beta_1)$, $w_k(\beta_2)$ be the volumes of the normalized eigenfunctions of (4.2) on the spherical segments Σ_1, Σ_2, respectively. It is not hard to show that

$$v_k(\beta_1)=R_1g_k(\beta_1),\ w_k(\beta_2)=R_2g_k(\beta_2) \tag{4.8}$$

To be specific we shall everywhere assume that $\beta_1 > \pi/2$, $\beta_2 < \pi/2$ for type-III equilibrium forms. We fix a certain angle β_1. To find the angle $\beta_2 < \pi/2$ for which the system $\{\Sigma_1, \Sigma_2\}$ loses stability we must solve (2.5) for $\lambda = 0$. This equation takes the following form when we allow for (4.4), (4.8):

$$\sum_{k=1}^m\frac{g_k^2(\beta_1)}{\eta_k(\beta_1)}+\zeta\sum_{k=1}^m\frac{g_k^2(\beta_2)}{\eta_k(\beta_2)}=0 \tag{4.9}$$

TABLE 1

m	β_1							
	91°	100°	115°	125°	135°	150°	160°	170°
1	89.0672	84.329	82.624	83.240	84.532	86.983	88.511	89.600
2	89.0659	84.272	82.467	83.045	84.330	86.832	88.425	89.576
3	89.0657	84.264	82.447	83.021	84.307	86.818	88.424	89.575
4	89.0656	84.262	82.441	83.014	84.301	86.815	88.423	89.575

where m is a sufficiently large integer, and

$$\zeta = \sigma_1 R_2^4 / \sigma_2 R_1^4 \tag{4.10}$$

Assume that a root of (4.9) has been found. We represent it by $\beta_2^\zeta(\beta_1)$. For any $\gamma \in (0, \pi/2)$ the inequalities [7]

$$0 < \eta_1(\gamma) < \eta_2(\gamma) < \ldots, \quad g_1^2(\gamma) < g_2^2(\gamma) < \ldots \tag{4.11}$$

are satisfied; it therefore follows from the form of (2.5), (4.9) that the system of menisci $\{\Sigma_1, \Sigma_2\}$ will be stable for a given ζ and angle β_1 if $\beta_2 < \beta_2^\zeta(\beta_1)$, and unstable if $\beta_2 > \beta_2^\zeta(\beta_1)$.

With the parameter ζ fixed, Eq. (4.9) determines a certain curve $\beta_2^\zeta(\gamma)$, $\gamma \in (\pi/2, \pi)$. The function β_2^ζ is continuous on the interval $(\pi/2, \pi)$, where $\beta_2^\zeta(\pi/2) = \beta_2^\zeta(\pi) = \pi/2$. We may use (4.11) to show that for $0 < \xi < \zeta$, the β_2^ζ curve is strictly below the β_2^ξ curve, i.e., $\beta_2^\zeta(\gamma) < \beta_2^\xi(\gamma)$, $\pi/2 < \gamma < \pi$.

Figure 3 shows the functions $\beta_2^\zeta(\beta_1)$ for several values of the parameter ζ (curves 1-10 correspond to ζ = 0.4, 1, 2.5, 5, 10, 20, 50, 100, 200, 1000). The values of the angles β_1, β_2 are plotted along the axes in degrees. According to the previous discussion, for any fixed value of $\zeta > 0$ the β_2^ζ curve divides the square $\{\pi/2 < \beta_1 < \pi,\ 0 < \beta_2 < \pi/2\}$ into a region of stable equilibrium forms (type III) and a region of unstable forms. Specifically, the part of the square below the β_2^ζ curve is the stability region and the part above, the instability region. The hatched region of Fig. 3 corresponds to instability of type-III equilibrium forms for $\zeta = 10$.

A digital computer was employed in the construction of the β_2^ζ functions. The eigenvalues $\eta_k(\gamma)$ and volumes $g_k(\gamma)$ (4.7) of the normalized eigenfunctions of (4.3) were tabulated for numbers k = 1, 2, ..., 6 and angles γ from 1 to 178°, for every degree. Values at intermediate points were found by interpolation; then for a broad set of parameters ζ we successively solved (4.9) for β_1 = 91, 92, ..., 178°; the number of terms kept varied from 1 to 6. Table 1 shows values of the roots $\beta^\zeta_{2,m}$ of (4.9) with $\zeta = 1$ for several angles β_1 and a number of terms m = 1, 2, 3, 4. As we see, the calculation process for $\beta_2^\zeta(\beta_1)$ converges well.

As we noted in §2, there may be cases of systems of two equilibrium surfaces that are stable under perturbations having zero volumes at each surface and unstable under arbitrary perturbations $N_1 \in D_1$, $N_2 \in D_2$, that are associated by the condition (1.10). The calculations show that one example of such a system is an arbitrary unstable form of equilibrium for the pairs of spherical menisci considered above [the first roots of (2.9) are positive for any angles β_1, β_2].

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